**Quantum Hall Effect**

So typically, the small B, large T situation is well described by the classical analysis. But in 2D, for larger B, smaller T, we see obvious evidence of quantization plateaus for ρxy and or oscillations for ρxx (this is similar to the magnetization oscillations we found in the Free electron magnetization file). This effect isn’t as prominent in 3D due to the continuous nature of the energy spectrum in the z-direction. But we’ll get to that in due course. Let’s review. So classically (see Classic Hall conductivity), our results for ρxx and ρyx,xy were:



if we plot ρxy and ρxx vs. B, we’d expect to find these two lines.

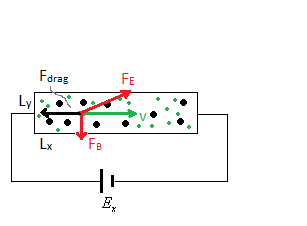
Chart, line chart

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So let’s see if we can explain this from a quantum perspective.

**Particle in magnetic field (Landau gauge)**

We can recover these results with a facile quantum calculation. Well not going to bother for ρxx, I’ll just pesume we get the same result as we did in the classical calculation ‘cause I don’t want to run through a Boltzman equation calculation with a magnetic field in it. But probably we’d get the classical result, with the added bonus of an explicit formula for τ. Apropos ρxy, things are easier, since the scattering time, τ, doesn’t even come into play, at least at the classical level. And so this doesn’t really have anything to do with statistical mechanics at all, just classical/quantum mechanics. Further, since the current in y direction, produced by the field in the electric field, Ex, in the x direction, doesn’t involve scattering, it is a thermal equilibrium phenomenon. So we can consider the Ex to be not a non-equilibrium force per se´, but part of the thermal equilibrium potential. So let’s consider the Hall setup again. We apply a magnetic field in the z direction, and a horizontal electric field, Ex, in the x direction. Imagining we’re dealing with positive charges for a second,



charge accumulates on bottom, creating an upward Ey field, which balances the downward force exerted by B. And so the charges’ velocity levels out into the x direction. And then we can measure the Hall coefficient via the transverse resistivity formula:



In our quantum calculation though, we don’t (or won’t) have an Ex *and* Ey field, just an Ex field. And we’ll be calculating the consequent jy current. We can (see Classical Hall Conductivity file again) measure *something* under these conditions.



ρyx and σyx are not necessarily reciprocals of each other. But it does turn out that (see Classical Hall Conductivity file again again) ρyx = σyx in the limit where τ→∞. So I guess we can hope to get ρyx this way. So let’s review the quantum states in a magnetic field in **k**-direction + electric field in x-direction (see Excitations file). So keeping our Landau gauge: **A** = (0, Bx, 0), our Schrodinger equation came to:



[e carries sign] Then we still have HO oscillator states, ultimately. Then using the ansatz, ψ = exp(ikyy)ψ(x), we found:



and so came to:



and,



So we saw that the degeneracy of the states was lifted, there being a term proportional to ky, which was ultimately proportional to xc = ky/eB + mE/eB2. Energy diagram (w/o spin) looks something like (for negative charges e < 0) below (two Landau levels depicted):

Chart, line chart

Description automatically generated

And we argued that the degeneracy (or really, Landau level *capacity*, now), drift velocity were:



and from this we can get the current density jy = Iy/Ly = nevy (see Classical Hall Effect for this definition)



where this N is total number of electrons, and n is charge density. There is another way to write this that will become relevant shortly. Consider the current Iy = jyLx supplied by a single Landau level.



So then the *current density* supplied by a Landau level is:



So interestingly, the current supplied by a Landau level is the same, regardless of the ambient magnetic field. And so the *total* current density would be:



where ν is the filling factor, the number of filled Landau levels (can be partial, like 3.2 or whatever). The filling factor would just be the number of particles divided by the degeneracy of the Landau levels:



This helps us understand that the resistance goes up as we increase B because the number of occupied Landau levels goes down, and so the current goes down. Of course our two expressions for jy are equivalent. Just to check:



So either way, filling this into the most relevant definition of the Hall resistance (see Classical Hall Effect file), we have:



and so, via our arguments above, we can conclude:



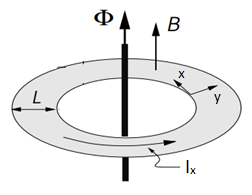
just as we found before in the classical case. This is what it looks like, more or less, in 3D, but not in 2D. We’ll get to why not in a bit.

**Particle in B field (Corbino Disk)**

Here’s another interesting way to look at this using a cylindrically symmetric setup. So we’ll look at a particle in a metallic disk, with an upward, homoegeneous, magnetic field B permeating the disk, and in addition a magnetic field generating some flux Φ, threading the center of the disk, but not extending into its interior. Going to work out the excited states, and then especially the ground state, and see what happens when we increase the flux Φ through that center thing. By Faraday’s law, this will induce a circular electric field through the ring (in the ‘-x’ direction), but then we’ll see it also pulls electrons radially inward at the same time, which constitutes a jy current. And from this we can get the Hall conductivity again via (see Classical Hall conductivity file):



So, starting out,



We’ll do the WKB approach again. So our H is:



Now though **B** is cylindrically symmetric, it isn’t constant everywhere, so we cannot say, per se´, that:



But we can use a little EM to work it out. For any circular loop within the disk we have:



where ΦB is whatever magnetic flux is. Well it’s ΦB = ΦB(center) + Bπr2, where ΦB(center) is the central flux, and B is the homogeneous field permeating the interior and the disk. Well, I’m going to call ΦB(center) as just ΦB, for concision. And we’ll say:



Projecting the eigenvalue equation onto the position/spin basis, then we’ll have the equation:



Let’s fill in the (2D) cylindrical coordinates differential operators, and our **A**.



So then,



We can postulate, as we did, that ψ = ei(m\_ℓ)φR(r), in which case we’ll get:



And putting in WKB form,



where,



This is of the same form as the example we did in the first Magnetic Field file in the Excitations folder,



So we can jump to that file’s conclusion,



and say (note in particular that ℓ´B = ℓB since our C2 = 0) that in our case we have:



But note in particular that α simplifies to:



So we have:



Will note the presence of the flux quantum Φ0 = 2π/|e|. And last for our purposes, we’d like to specialize to the GS and consider where the peak of the wavefunction resides. So we’ll go to η = 1, mℓ = whatever is necessary to further minimize the energy [well, this would mℓ = (0,1,2,3,…,∞)·sgn(e)]. Then copying our result from the previous file,



we’ll have:



So one thing we can see is that if we increase ΦB, then α decreases (because if e is negative, say, then as we argued a paragraph above mℓ is presumed negative for Ground States, and -ΦB/(2π/e) will be positive, and so α = mℓ - ΦB/(2π/e) will get smaller; likewise for positive values of e). This means that rpeak decreases as well. And in fact if we increase it by one flux quantum ΔΦB = Φ0 = 2π/|e|, then α will decrease by 1. This is consistent with what we might expect classically, since if we increase ΦB, this will, by Faraday’s law set up a CW electric field, which will exert a CCW force on negatively charged electrons, which will slow them down since mℓ < 0 means they were traveling CW to begin with, and so having been slowed, their radius in the magnetic field B will shrink. This shrinkage is:



And so the change in radius is about,



Multiplying by e, and dividing both sides by δt, we’d have:



Now δrpeak/δt is the velocity of our charge in the ‘y’ direction. And we can recognize δΦB/δt as the induced emf, from Faraday’s law, and further, (δΦB/δt)/2πrpeak as the induced electric field, Ex, in the vicinity of our charge. So we have:



Multiplying both sides by the charge density,



And so we get what we wanted:



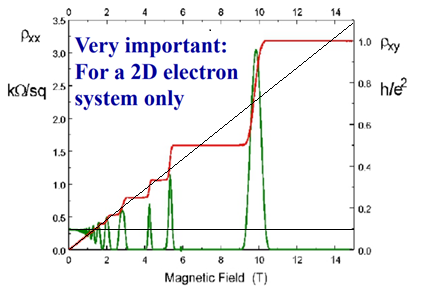
and so once again we conclude:



Again, this is kind of what it ends up looking like in 3D, but not in 2D. Now let’s get into why.

**Actual Results**

So it turns out this is not what we find experimentally, 2D. Rather we see something like the red (ρxy) and green (ρxx) curves:



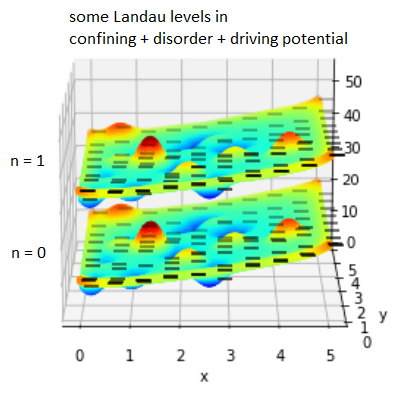
It won’t be too hard to see why ρxy should be that way at least. But we’ll have to take into account an explicitly 2D model, and include within the electric potential the confining potential, the disorder potential, and the driving electric field potential set up by the battery.



We can’t explicitly solve this model, but we can argue, like we did in the excitations file, that for small potentials/large B’s, we’ll still have distinct Landau levels which will follow the contours of the potential. So we’ll have:



Pictorally, the energy levels look something like this (the upward slant is due to Φbattery = -eEx, and they’d be tilted downwards rather, for positive charges):



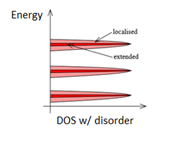
The ‘degeneracy’ and drift velocity of the levels would be as usual (ignoring spin),



So as we noted in the Excitations file, all those particles up there in the graph traverse the equipotential they’re on, which makes those on the upper potential (red) or lower potential (blue) localized, and therefore unable to contribute to the current [okay technically, with Φbattery = -Ex added to Φ, what used to be an equipotential is now not, but as we’ll argue at end, it is really not legitimate to consider Φbattery as *truly* part of the potential anyway). Only the particles on the middle (green-ish) potential would be in extended states, and therefore able to contribute. So since not all particles contribute to the current, we can’t calculate the current the way we did in the first example,



So what is the current contributed by particles occupying the green edge states, i.e., the dark red states represented in the density of states diagram below?



We’ll look at a single Landau level with a filled extended states part. So first, we have:



Now they argue that we can in fact extend the sum over all states, regardless of whether the states are extended or not. This is because we’ll just get zero for localized states, as these are characterized by lying on an equipotential and so φ won’t be changing for them, and so when we integrate from one end of the equipotential to the other, we’ll just get zero, as can see that integrating ∂φ/∂x between two points of equal potential will give zero. That doesn’t make sense because while φ is constant, that doesn’t mean ∂φ/∂x = 0, right? Maybe we can just say that for the localized states, Ex will be going one way often as the opposite way (since it’s a closed loop), and so it averages to zero? There are better arguments out there, but that’ll do for me. So now we can say,



So as long as the extended states are filled we have the same current per Landau level as before in the first example. So then to get the total current, we just multiply by the number of Landau levels with filled extended states. This would be the number of levels that are at least half-filled. Guess I’ll write it like this:



And ν>1/2 would be:



where [x] is the floor function, giving the greatest integer less than x. We can simplify our result a bit more. From one end of the sample to the other, the net change in potential would be ΔΦx = -eExLx. So filling that in,



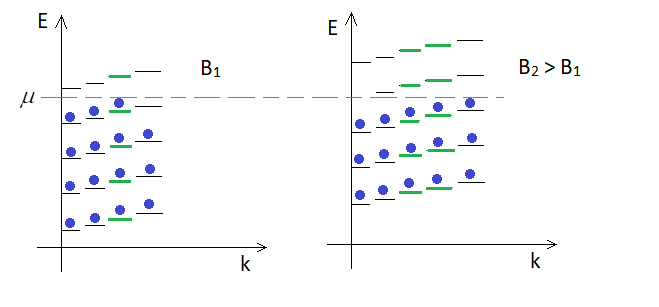
And then we’d have:



and so, via our arguments at the top of the page, we can conclude:



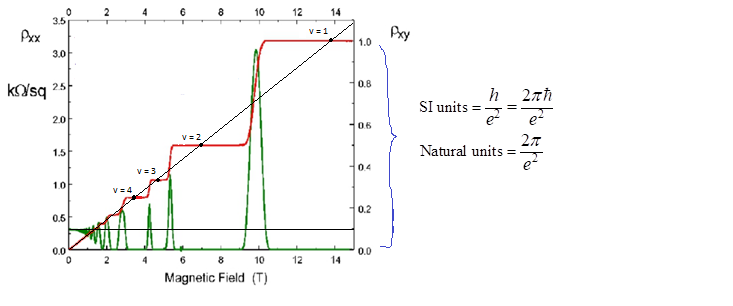
and this is where we get the plateaux from. So every time there is a jump in ρxy, the present topmost Landau level’s set of delocalized states has been emptied (with increasing B, the degeneracy of the Landau levels increases and so particles will be continually depopulating the highest Landau level and dropping into lower ones). In other words, at every jump in ρxy, the new top most Landau level has become less than ½ filled. And ρxy will be constant until the next lower Landau level becomes the topmost, and its extended states are depopulated. Kind of illustrated here, though just in 1D:



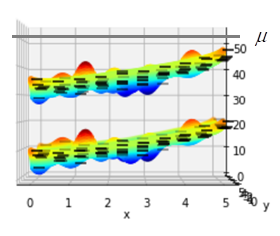
(extended states are in green, and I don’t really know if number of extended states increases with degeneracy but it doesn’t matter that much either way) So at the center of our plateaux, the Landau levels are completely filled. Now [x-1/2] = x at center of the plateaux. And so at the center of the plateaux, ρxy would be given by:



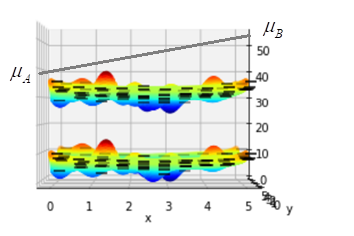
which is the classical value. And this just what we see in that experimental plot as the classical line intersects the midpoints of the plateaux. Gonna label the ν’s on our graph just to make more explicitly clear.



(note h/e2 ≈ 25.8kΩ) Okay, one more gripe about our calculation is that in reality, the driving field, Ex is not an ‘equilibrium’ field, and so Φbattery = -eEx doesn’t really belong as part of H, at least in so far as determining the allowed energy values of our particles. So it isn’t really correct to depict the levels slanted by Φbattery = -eEx, like shown below, and filled up to the chemical potential (well we just fill up to however far we get with N particles, not really a chemical potential, but you know)



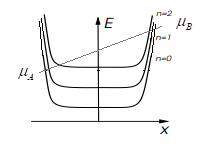
Rather the levels are horizontal, and the chemical potential at either side of the sample is different thanks to the same driving electric field,



where line going from μA to μB would be given by –eEx. The slope of the line is definitely exaggerated in comparison to the Landau level. Now we can still use our formula for the current density for each Landau level,



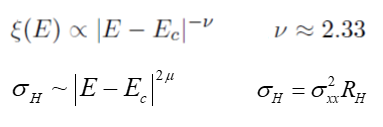
Then remembering that the confining potential actually makes the potential profile of each Landa level go up to ∞ on the edges, we can see that ΔΦx is still -eELx for each level. This is depicted by a picture I found in one author’s paper.



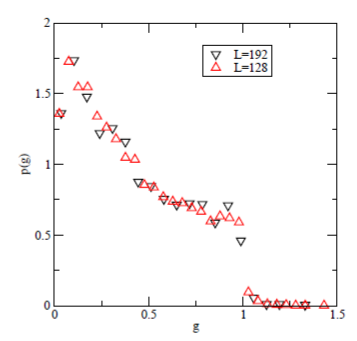
Last, what about ρxx? We see that it’s non-zero only when ρxy is transitioning from one plateau to another. And at this time, the delocalized states in the uppermost Landau level are depopulating. I would think that it’s only during this time that the charges have states they can back scatter into, and so the resistance is finite. But when the delocalized states are filled, there can be no scattering (unless there are nearby extended states in the bulk), and so the resistance drops to zero.

**Is this strictly 3D? or some different symmetry class?**

There is evidently a phase transition here, when the magnetic field is strong enough to keep the Landau levels separate despite the broadening effect of disorder? In that case, the localization length goes as:



Evidently the Hall conductance doesn’t acquire any corrections in weak field, which seems to be like what happens for conductance in general with magnetic fields? And Markos presents P(g) at the critical point. It seems we get pretty much the same behavior regardless of the Landau band we’re in…I guess that’s reasonable as no one asks which band we’re in for the other cases.



So I thought all 2D states were localized. Is this only for orthogonal case? Or maybe they are, but ξdisorder is so large that it doesn’t make a practical difference. Or maybe broken TRS but strong B? If so, how strong? The graph on previous page would seem to indicate we have conduction for arbitrary B. MacKinnon review does seem to suggest that ρxx and ρxy (or ρH) are separate scaling variables – so have 2 parameter theory. But then elsewhere he says SPS was demonstrated, numerically. It’s possible we’re talking about a 3D system though.

Later he says that there are two phase transitions: increasing B will drive a transition from insulator to metal, and then eventually, back again, with even stronger B.

P. Lee says there aren’t any localization corrections to RH to first order at least.